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Basics of Second-Order Predicate Logic

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Abstract

The article presents the basics of second-order predicate logic (SOL). The need for a symbolic representation of the general quantifier is pointed out. A distinction is made between the first-order predicate logic (FOL) and the second-order predicate logic (predicates of predicates, relations of relations). The syntax and semantics of the second-order predicate logic are introduced. Logical and non-logical designators and operators, terms, rules for forming logical formulas, status of variables, and rules for variable substitution are introduced. Reference is made to Henkin's semantics of controlled predicates, and an axiomatic system of second-order predicate logic. Russell's analogy for the axiom of choice and methods of proving the validity of the deduction for second-order predicate logic are given.

Key Words: logic, symbolic logic, predicate calculus, second-order predicate logic

Introduction

Generalized predicate logic or second-order predicate logic (SOL) is an extended formal logical-symbolic and deductive system of first-order logic (FOL), which is itself an extended formal and deductive system of propositional logic (PL). Second-order predicate logic (SOL) includes as objects of the domain or universe of discourse

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those objects that can be quantified functions and sets of sets, i.e. predicates of predicates and relations of relations.

What cannot be expressed by the first-order logic and why is it necessary to extend the limited first-order logic with a generalizing quantifier? According to Jouko Väänänen (2011, 283) first-order logic, if limited to finite semantic and deductive models, is not capable of expressing the following statements

" There are infinitely many *x*'s such that........"
"There is an even number *x* such that........"

These are examples of new logical operations called GENERAL QUANTIFIERS. In our natural language, according to Väänänen (Ibid, 284), there are a large number of generalizing quantifiers, e.g.

Two thirds voted for John.
Exactly half remains.
Most wanted to leave.
Some but not all liked it.
Between 10% and 20% were students.
Hardly anyone touched the cake.
The number of white balls **is even**.
There are infinitely many prime numbers.
There are countless things.

First-order predicate logic can be extended by adding such new quantifiers. In case (1) "There are infinitely many *x*'s such that........" the resulting logic cannot be axiomatized, but in the form (2) "There are uncountably many *x*'s such that..... ...." it is possible to axiomatize a new logic (second-order predicate logic).

Therefore the SOL is a formal logical system (symbolic language, according Rudolf Carnap – *Introduction to Symolic Logic and Its Applications, 1958*) that has its own syntax, semantics, and proof deduction theory that is given in mathematical metatheory (most often set theory). Second-order logic (SOL) and first-order logic (FOL) was introduced by the German mathematician and logician Gottlob Frege in his *Begriffsschrift, eine der arithmeticen nachgebildete Formelsprache des reinen Denkens*. Halle, 1879 (Concept script, the language of formulas of pure thought made according to the language of arithmetic).

Gottlob Frege created a form of calculation by extension of terms of the second degree (*Begriffswort*) where the basic logical relation is SUBORDINATION OF CONCEPTS (predicates of predicates), in contrast to first order logic which has a form of calculation by extension of individual terms (Begriff) and where the basic logical relation is SUBSUMATION OF OBJECTS UNDER A CONCEPT (predicates of individual objects / concepts).

Quine's claim from *Philosophy of Logic* (1970, 66) is well known that SOL is disguised set theory, i.e. "set theory in sheep's clothing ", considering that set theory has an "amazing ontology" (object-bound variables) while logic should be taken without ontology (virtual objects). It was Quine in *Philosophy of Logic* who abandoned the set theory as the basis of symbolic logic, rejected the concept of an object-bound variable and advocated the concept of virtual classes and sets as variables. There is also Quine's remark that SOL is "much more mathematics than logic" and that it presupposes a lot of mathematical knowledge. The SOL has become the logic programming language for computer science.

Proponents of the opinion that SOL is logic and not mathematics believe that the syntax of SOL with standard semantics is sufficiently clear, intuitive and unproblematic that it can express the appropriate framework for axiomatization and the principles of mathematics (cf. Shapiro, 1991, p.204). According to Fisk (1964, p.63) a quantification formula (or a "predicate schema" according to Carnap and Quine) "is an expression that contains one or more free variables and which becomes a statement when all variables are either eliminated or substituted." The difference between a monadic and a polyadic predicate variable is written by the formula 'fx' and 'fxx' or '$fxyz$' and is a possible ka n-adic variable.

If the quantification formula contains only one variable (eg x) on which the universal and existential quantifier is applied, then it is called a monadic predicate formula; if it contains several variables (x, y, z,..) to which quantification is applied, then it is called a polyadic predicate formula. Monadic quantification of predicates or 1st-order logic is so called because quantification is performed only on one variable, that is, universal existential quantifiers refer only to the property or properties attributed to one individual thing (object) in the domain or universe of discourse that is denoted by the variable (cf. Fisk, 1964).

Second-order logic is "what becomes first-order logic when we allow the universal and existential quantification of predicate letters" (cf. Jeffrey & Burgess, 2006, p.125), that is, the quantification of predicate variables of first-order logic, as in the example where on the quantification of a one-place predicate P attributed to two individual things denoted by variables, eg x and y.

This means that in the SOL it is a polyadic predicate formula.

An example given by Jeffrey and Burgess (2006, p.127)

$$\forall x\, \forall y\, [\, x = y \leftrightarrow \forall P\, (\, Px \rightarrow Py\,)]$$

shows the predicate formula derived from the identity axiom extended by (1) an additional variable, (2) a universal quantifier, and (3) an identity sign. It is a classic example of a predicate formula or predicate scheme of SOL.

In the case where it is a two-place predicate that expresses the relation (R) between any two individual objects, for example x and y, as follows

$$\forall x\, \forall y\, \exists R\, Rxy$$

which should be read in the meaning: "For each x and for each y there is some relation R such that each x stands in the relation R with each y."

Here, the second-order predicate formula is extended by an existential quantifier, a universal quantifier, another variable, and a relation.

In SOL, the quantification of declarative functions, that is, symbols for functions, is performed as in type generalization

$$\exists z\, \exists f\, \forall P\, \{\, [\, Pz \wedge \forall x\, (Px \rightarrow Pfx)] \rightarrow \forall x\, Px\, \}$$

Syntax of the language of SOL

Formal languages that contain variables that extend to elements of the universe of discourse such as properties, functions, sets, relations, which by definition are found as elements of a certain universe of discourse, represent the language of SOL.Variables that extend to such objects (which are themselves represented by variables) are called SECOND-ORDER VARIABLES.

As in the logic of statements and in the monadic logic of first-order predicates, language and logic are mutually determined through logical syntax and logical semantics.

According to Shapiro, a language "that contains first-order variables and second-order variables, and no others, is a second-order language with a focus on second-order logic". (cf. Shapiro, 1991, p.v)

The syntax of the language of SOL does not expand the number of illogical symbols used to describe constants, that is, designators that exist in first-order logic. The syntax is determined by one finite set of countable symbols. The interpretation is also determined by the domain and the function assigned to the objects as elements of the domain.

What is extended in relation to FOL are logical variables that are used to designate relations of relations, functions on relations, functions on functions, etc. This means that this process of reduplication of quantification objects within a domain can be repeated through higher-order logic than SOL by introducing third-order or fourth-order variables, etc.

Logical and Non-logical Designators

ELEMENTS OF LANGUAGE	SYMBOLS
Designators for predicates with indication of locality if necessary	$A, B, C, \ldots, Z$ $A^1, B^1, Z^1, A^2, A^3, \ldots Z^3, \ldots$
Designators for constants with a subscript if necessary	$a, b, c, \ldots, w$ $a_1, w_4, h_7, m_{32}, \ldots$
Designators for atomic formulas of second-order logic	Φ, Ψ
Designators for individual variables with a subscript if necessary	$x, y, z, \ldots$ $x_1, y_1, z_1, x_2, \ldots$
Variable designators for properties with a subscript if necessary	$X, Y, Z, \ldots$ $X_1, Y_1, Z_2, \ldots$
Variable designators for relations with a subscript if necessary	$R, S, T, \ldots$ $R_1, R_2, \ldots S_1, T_2, \ldots$
Variable designators for functions with a subscript if necessary	$F, G, H, \ldots U, \ldots$ $F_1, G_2, H_3, \ldots U_3$
Term designators (constants and individual variables)	t $t_1, \ldots t_n$
Designators for logical connectives	$\neg, \wedge, \vee, \supset, \equiv$
Logical operation scope indicators	(,), [,], { , }
Designator for universal quantifier	$\forall$
Designator for existential quantifier	$\exists$
Designator for UD (universe of discourse)	$U, V, \ldots$
Designator for a logical structure or logical model	M

Terms

In second-order predicate logic, the term TERM denotes a general term for the ARGUMENT of a certain logical operation in which the predicate or function or relation appears. The role of TERMA is defined according to the following rules:

I. Every individual variable and every constant designator is a term.

II. If there is an n-place function variable f^n in the formula and if the formula contains n number of arguments, i.e. if $t_1, t_2, \ldots t_n$ terms, then the formula $f^n(t_1, t_2, \ldots t_n)$ is also a term (thus an argument for a new higher order of predication).

Rules for Formula Formation

In SOL, as in any formal logical system, there are certain rules for the construction of a complex second-order predicate formula from a second-order atomic formula.

I. Every atomic formula is a formula.

II. If the formula contains a relational designator for the relational variable R that appears in n-places in the formula and if arguments, i.e., terms t in n-places appear in the formula as a sequence $(t_1, \ldots, t_n)$, then $R(t_1, \ldots., t_n)$ atomic formula.

III. If f is a one-place function variable and if the term $t(t_1, \ldots., t_n)$ is a sequence of terms, then the formula $t(t_1, \ldots., t_n)$ is an atomic formula.

IV. If A and B are SOL formulas, then $\neg A, A \wedge B, A \vee B, A \rightarrow B, A \leftrightarrow B$ are also SOL formulas.

V. If A is a SOL formula, and x is an individual variable, then the quantified schemes of the formula A by the individual variable x are also formulas: $\forall x A, \exists x A$.

VI. If A is a SOL formula and R is a relational variable, then the quantification schemes of the formula A by the relational variable R are also SOL formulas: $\forall RA, \exists RA$.

VII. If A is a SOL formula and f is a function variable, then the quantification schemes of the formula A by the function variable f are also SOL formulas: $\forall fA, \exists fA$.

VIII. Logical constants ⊤ and ⊥ are atomic formulas.

Status varijables

All variables in the atomic formula A in SOL are free, that is, the occurrence of a variable in the atomic formula is not bound by quantifiers.

In quantified formulas of the type $\forall xA, \forall RA, \forall fA$, each occurrence of the individual variable x, the relational variable R, and the functional variable f is bound by a quantifier.

If the variables x, R, or f have a free occurrence (or if they are bound by a quantifier) in the atomic formula A, then so is their occurrence in the formulas $\neg A, A \wedge B, B \wedge A, A \vee B, B \vee A, A \rightarrow B, B \rightarrow A, A \leftrightarrow B, B \leftrightarrow A, \forall y A, \exists y A$ where B is an arbitrary formula and y is a variable that differs from the variables x, R, and f.

In atomic formula A, a variable is free if it appears or if it occurs at least once unbound by a quantifier.

If the formula has no free variables, then it is closed ("saturated", "filled" in Frege's terms) and then it is not a statement function but a statement.

Rule for Substitution of Variables

In the atomic formula A, in the process of proving validity, we can replace the individual variable x with the term (argument) t in each of its occurrences or occurrences as a free variable, which is expressed in the form $A\ (t\ /\ x)\ or\ A\ (t\ )$.

Standard Semantics of the SOL

The SOL is more intuitive and expressive than FOL and less formalizable, which means it is less secure and less complete than FOL. Second-order logic more closely expresses informal ways of thinking and is closer to the way of thinking and expressing in everyday language. Let's look at the statements:

"Fox caught the chicken. " (1)

"**One (some)** fox caught all the chickens. " (2)

"**All** the fox have caught all the chickens. " (3)

In statement (1), we do not know, if we do not assume or if we do not know from earlier discourse, whether it is about a (one) fox or about every fox. In statement (2), one variable is determined by the quantifier. If one wants to express with predicate logic that one fox caught all the chickens, then a formula is obtained in which two different quantifiers participate

$$\exists x\ \forall y\ [\text{fox}\ (x) \wedge [\ \text{chicken}\ (y)\ \rightarrow\ \text{caught}\ (x, y)]]$$

$$\exists x\ \forall y\ [P\ (x) \wedge [\ Z\ (y)\ \rightarrow\ R\ (x, y)]]$$

In statement (3), which is formulated in the manner of SOL, we have the relation of two sets of predicates or two sets or two universes of discourse "all fox" and "all chickens ". We quantify all variables with a universal quantifier

$$\forall x\ \forall y\ [\text{fox}\ (x)\ \rightarrow\ [\ \text{chicken}\ (y)\ \rightarrow\ \text{caught}\ (x, y)]]$$

This formula can be read as follows: "For all x's and for all y's it is true that, if x is a fox, then it is true that, if y is a chicken, then it is true that x caught y".

If the designators of the predicate logic of the second order are applied, then the complete formula is obtained

$$\forall x\ \forall y\ [P\ (x)\ \rightarrow\ [\ Z\ (y)\ \rightarrow\ R\ (x, y)]]$$

Henkin's Semantics of Controlled Predicates

In addition to the standard semantics of SOL, which is also called complete semantics, there is also Henkin's semantics.

Henkin's semantics (named after the semantic system of controlling predicates proposed by the American logician Leon Albert Henken, professor from Berkeley and associate of A. Tarski) has several elements of the semantics of FOL.

The scope of the quantifier is not open to all structures that can be constructed in one semantic model, but is limited and determined.

It does not include all sets of functions or all predicates of predicates. Henkin's semantics strategy introduces a restriction on the types of predicates and operates, similar to FOL, only with defined predicates, that is, with predicates that can be defined. (Cf. Manzano, M., Sain, I., Alonso, E. (eds. 2014)

Axiomatic System of SOL

The SOL is not as reliable and complete as FOL, and classical deductive proof cannot be performed in it.

Its real quantification possibilities are limited to one discourse universe with functions, relations, individual variables, and its exaggerated expressiveness or syntax gives the image as if it can quantify all variables that include all subsets of all discourse universes, which is not possible by itself.

This is why more Henkin semantics are applied to proof deduction than standard SOL semantics.

According to Hilbert and Ackermann, if the predicate variables enter the axioms that are in the atomic formulas of the predicate calculus, then it is a second-order axiomatic system (Hilbert, D., Ackermann, 1950, p.107). Such a predicate calculus is called a spread predicate calculus.

The axioms of quantification and the rules of deduction of evidence that are set in the standard semantics of SOL are actually extended axioms of FOL to second-order variables (cf. Shapiro, 1991, 66). According to Hilbert and Ackermann in *Grundzüge der tehoretischen Logik* (1967, 144) the framework for the axiomatic system of SOL is already given in FOL (cited: 79 – 83)

1) $\forall x\,(Fx \vee \neg Fx)$
2) $\forall x\,Fx \supset \exists x\,Fx$
3) $\forall x\,(A \vee Fx) \supset A \vee \forall x\,Fx$
4) $\forall x\,(A \supset Fx) \supset (A \supset \forall x\,Fx)$
5) $A \supset \forall x\,(A \vee Fx)$
6) $\forall x\,(A \vee Fx) \equiv A \vee \forall x\,Fx$
7) $\forall x\,Fx \supset Fy$
8) $F\,y \supset \exists\,x\,Fx$
9) $\forall x\,(Fx \supset A) \supset (\exists\,x\,Fx \supset A)$
10) $\forall x\,(A \wedge Fx) \equiv A \wedge \forall x\,Fx$
11) $\exists\,x\,\exists\,y\,Fxy \equiv \exists\,y\,\exists\,x\,Fxy$
12) $\forall x\,\forall y\,Fxy \supset \forall y\,\forall x\,Fxy$
13) $\forall x\,(Fx \wedge Gx) \equiv \forall x\,Fx \wedge \forall x\,Gx$
14) $\forall x\,(Fx \supset Gx) \supset (\forall x\,Fx \supset \forall x\,Gx)$
15) $\forall x\,(Fx \equiv Gx) \supset (\forall x\,Fx \equiv \forall x\,Gx)$
16) $\exists\,x\,Fx \equiv \neg\forall x\,\neg Fx$
17) $\exists\,x\,\neg Fx \equiv \neg\forall x\,Fx$
18) $\neg\exists\,x\,\neg Fx \equiv \forall x\,Fx$
19) $\neg\exists\,x\,Fx \equiv \forall x\,\neg Fx$
20) $\forall x\,(Fx \supset Gx) \supset (\exists\,x\,Fx \supset \exists\,x\,Gx)$
21) $\forall x\,(Fx \equiv Gx) \supset (\exists\,x\,Fx \equiv \exists x\,Gx)$
22) $\exists\,x\,\forall y\,Fxy \supset \forall x\,\exists\,y\,Fxy$
23) $\forall x\,\forall y\,Fxy \supset \forall x\,Fxx$
24) $\exists\,x\,(A \wedge Fx) \equiv A \wedge \exists\,x\,Fx$
25) $\exists\,x\,(A \vee Fx) \equiv A \vee \exists\,x\,Fx$
26) $\exists\,x\,(Fx \vee Gx) \equiv \exists\,x\,Fx \vee \exists\,x\,Gx$
27) $\exists\,x\,(Fx \vee A) \equiv \exists\,x\,Fx \vee A$
28) $\forall x\,(Fx \vee A) \equiv \forall x\,Fx \vee A$
29) $\exists\,x\,Fxx \supset \exists\,x\,\exists\,y\,Fxy$
30) $\exists\,x\,\neg\exists\,y\,\neg(Fx \vee \neg Fy)$

This system of axioms is supplemented with an AXIOM SCHEME OF COMPREHENSION (ABSTRACTION) for a quantified relation or an individual variable of the second order or a function in SOL.

If $\phi(x_1. \ldots . x_n)$ is formula in SOL with x1. . . . . xn by a series or sequence of free individual variables in it and if it contains a free relational variable R of the second order in itself, then the axiom scheme of comprehension can be represented by the following formula:

$$\exists R \, \forall x_1. \ldots . x_n \, (\phi(x_1. \ldots . x_n) \leftrightarrow R(x_1. \ldots . x_n)).$$

If it is a predicate variable, then the comprehension scheme has the form

$$(\exists X) \, [\, X(x_1. \ldots . x_n) \leftrightarrow \phi(x_1. \ldots . x_n) \,].$$

where $\phi(x_1. \ldots . x_n)$ is a formula containing a free predicate variable.

Such axioms, such as the axiom of comprehension, according to Mevlin Fitting (2002, p.3), ensure that each predicate variable corresponds to an "object". These axioms were supplemented with the AXIOM OF CHOICE, formulated by Ernst Zermelo in 1904, and also applied by Hilbert and Ackerman in the work *Grundzüge der tehoretischen Logik*.

The Axiom of Choice aims to define the concept of an ordered sequence in set theory more strictly than the axiomatic scheme of understanding allows. Hilbert and Ackermann connected it with the rule of separation (as sometimes called modus ponendo ponens) (Hilbert & Ackermann, 1967, 145)

The axiom of choice arose in relation to solving problems, that is, more than one rhetorical question: is every group a well-ordered group?

In 1883, George Cantor stated in his work "*Die Grundlagen eine allgemeine Manningfaltigkeitslehre*" that according to the laws of thought, any group can be well ordered. (cf. Cantor 1883a, 550. Gesammelte Abhandlungen (Cantor 1932).

In 1904, the German mathematician Ernst Zermelo was the first to formulate the axiom of choice. In order to defend the importance of introducing the axiom of choice, Zermelo created his own system of axiomatization in 1908.

According to G.H. MOORE [1982, p. 1-Prologue] "... the axiom of choice asserts that for every set S there is a function f that connects every nonempty subset A of the set S with a unique member $f(A)$ of A."

If the expression $A\ (x_1.\ .\ .\ .\ .\ x_n\ ,\ y\ )$ is a formula containing the free variables $x_1.\ .\ .\ .\ .$ x_n and y is then a formula

$$\forall\ (x_1.\ .\ .\ .\ .\ x_n\ \exists\ y\ A\ (x_1.\ .\ .\ .\ .\ x_n\ ,\ y\ ) \rightarrow \exists\ F\ [\ \forall\ (x_1.\ .\ .\ .\ .\ x_n\ \exists\ y\ (\ F\ x_1.\ .\ .\ .\ .\ x_n\ ,\ y \wedge A\ (x_1.\ .\ .\ .\ .\ x_n\ ,\ y)) \wedge\ \forall\ (x_1.\ .\ .\ .\ .\ x_n\ yz\ (F\ x_1.\ .\ .\ .\ .\ x_n\ ,\ y \wedge F\ x_1.\ .\ .\ .\ .\ x_n\ ,\ z \rightarrow y = z)]$$

is an atomic formula. At the same time, the expression y = y should be seen as an abbreviation of the expression $\forall\ G\ (Gy\ \rightarrow\ Gy)$.

According to David Hilbert, the meaning of this axiom is as follows:

If the formula $\forall\ (x_1.\ .\ .\ .\ .\ x_n\ \exists\ y\ A\ (x_1.\ .\ .\ .\ .\ x_n\ ,\ y\ )$ is correct, then each n-place individual variable $x_1.\ .\ .\ .\ .\ x_n$ has been assigned a certain value y with the property $A\ (x_1.\ .\ .\ .\ .\ x_n\ ,\ y\ )$.

According to S.SHAPIRO [1991, p.67], since the first axiom of choice is formulated as a conditional, then the antecedent of this conditional asserts that for every sequence ($x_1\ .\ .\ .\ x_n$) there is at least one y such that the sequence ($x_1\ .\ .\ .\ x_n\ ,\ y$) satisfies A. This means that y has x_n choices if there exists some choice function for every nonempty subset of y indexed by x.

For the FOL or for limited predicate calculus, it is possible to have a universally valid formula that can be deduced using certain rules from the stated axioms.

However, for SOL or for extended predicate calculus, it is not possible to make a complete axiomatic system for a universally valid formula of SOL.

Kurt Gödel showed that for every system of primitive (atomic) formulas and rules of inference, universally valid formulas can be made that are undecidable, that cannot be decided whether they are true or false.

Russell's Analogy for the Axiom of Choice

In the book *Introduction to mathematical Logic* (London, 1919) in chapter XII (*Selections and the Multiplicative Axiom*), Russell introduced an analogy by which he showed the problem concerning the axiom of selection by deriving it from the problem of applying the axiom of multiplication with an infinite and unselected number of factors.

Correct procedure in mathematical operations requires that some current class with a defined number of factors be constructed. With the addition operation, this is not a problem: if some cardinal number μ and a class α which has a μ term are given. The question is how to define the addition $\mu + \mu$?

Here, it is necessary to construct two classes from class α that will be ordered so that first a class is formed that contains ordered pairs whose first term is a class consisting of some singular member of class α and whose second term is the zero class.

After that, the second step is to form all ordered pairs whose first term is the zero class and whose second term is the class consisting of some singular member of the class α. These two classes of pairs have no common term and the logical sum of these two classes is $\mu + \mu$ term. Exactly such a procedure can be applied to proving $\mu + \nu$ if it is about two classes α and β, where α has μ members and β has ν members. However, the problem arises with multiplication where there is an infinite number of factors and where it is not possible to make a selection of members

Validity of Deductions

If, in accordance with Shapiro's (1991) concept of SOL, we denote the deductive system of FOL with the symbol D1, then we can denote the extended system of deduction for SOL with the symbol D2.

This expansion of system D1 concerns the introduction of new axioms and new rules of inference related to relational and functional variables that appear in SOL.

In addition to these two elements of the deductive system D2, it is necessary to take into account the conventions regarding the status of bound and free variables.

Examining the validity of SOL reasoning using the truth tree method in order to establish the truth of the assertion of the statement formula, which claims that one and the same predicate (property) belongs to two individual things, x and y, i.e. that they are therefore identical.

The second-order declarative formula asserts that x and y are identical objects if and only if the property P in its entire scope is covered by the same term (property, predicate).

The proof here is done by making a counter-factual example, i.e. by proving the opposite statement: that there is no identity of x and y, ie that there is some predicate P that does not belong either to some x or some y. If this is proven, then it means that there was a mistake in our assumption, otherwise, if this negative assumption is not proven, then the initial assumption was true.

1. $\forall x \, \forall y \, [\, x = y \leftrightarrow \, \forall P \, (\, Px \, \rightarrow Py)]$ $\qquad P$

2. $\neg \forall x \forall y \, [x = y \leftrightarrow \forall P \, (Px \rightarrow Py)]$ $\neg P$

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3. $\exists x \neg \forall y \, [x = y \leftrightarrow \forall P \, (Px \rightarrow Py)]$ (from 1.)
4. $\neg \forall y \, [a = y \leftrightarrow \forall P \, (Pa \rightarrow Py)]$ (from 2.)
5. $\exists y \neg [a = y \leftrightarrow \forall P \, (Pa \rightarrow Py)]$ (from 3.)
6. $\neg [a = b \leftrightarrow \forall P \, (Pa \rightarrow Pb)]$ (from 4.)
7. a = b ¬ a = b (from 5.)
8. $\neg \forall P \, (Pa \rightarrow Pb) \; \forall P \, (Pa \rightarrow Pb)$ (from 5.)
9. $\exists P \neg (Pa \rightarrow Pb)$ (from 7.)

Conclusion

The science of logic includes research into the functioning of various concepts of the deduction system that are constructed in relation to the goals and purposes to be achieved in a certain domain or universe of discourse through certain logical and linguistic practices. The question "Which logic is the right logic" always initiates consideration of the completeness of logical relations in the deductive system, that is, the strength and coherence of the syntactic and semantic consequences of reasoning in that system and their relations.

The development or improvement of deductive systems is realized by expanding the operations that are entered into the previous already existing deductive systems: first-order predicate logic (FOL) is created by expanding the operations and symbolic procedures of propositional logic (PL); second-order predicate logic (SOL) is created by extending the procedures of first-order predicate logic. All this is done with the intention of improving the character of the completeness of the deductive system.

There are doubts about SOL, whether it is a sufficiently complete system of deduction that can simultaneously express the generalization of constants and variables, whether it is proper logic or mathematics at all, whether it is applied in linguistic practices at all, natural language or only in computational practices of constructing an algorithmic structure ... Besides that, according to Marcus Rossberg (2004. pp.303-321) "recent criticisms focus both on the ontological commitment of SOL, which is believed to be to the set-theoretic hierarchy, and on the allegedly problematic epistemic status of the second-order consequence relation".

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